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Frege on Numbers

"Die Logik ist keine Lehre, sondern ein Spiegelbild der Welt."
*(Wittgenstein, **Tractatus**, 6.13)*

Introduction

Frege's philosophy of mathematics is generally seen as bearing two main labels: *Logicism* and *Platonism*. The aim of my paper is to present his conception of numbers as logical objects, and, consequently, to mainly discuss Frege's Platonism. The issue could be approached from two perspectives. On the one hand, we can ask *ontologically* what are numbers and where could they be found? On the other hand, we can ask *epistemologically* what could we know about numbers and how could we obtain such knowledge? Even is true that both points are connected, and it is very hard to make a sharp distinction between them, at least in Frege's case, I will try to pursue the first ontological approach.

No doubt, we can speak about a Fregean ontology, but an explicit exposition is not to be found in Frege's philosophy. One may only find some

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sort of implicit ontological doctrine, in the sense that we can deduce his ontological commitments from other clear claims. But this hidden ontology has at least two main explicit suppositions. The most fundamental one is the distinction between *object* and *concept*, a distinction that applies to every entity. It is clear that this could be seen as a kind of ontological axiom, which enables us to build an ontological system. The second main distinction is semantic, yet it generates some ontological constraints. This is the well-known distinction between 'sense' (*Sinn*) and 'reference' (*Bedeutung*). As Bergmann pointed out,[1] in ontological discourse two clusters of very ordinary words are used philosophically. One cluster contains 'thing', 'object', 'entity', 'existent', and so on. These terms belong to what we should call *formal ontology*, and the first distinction works mainly in this direction. The other cluster contains 'naming', 'denoting', 'designating', 'referring'... In this case, the route to our ontology is *semantic*, and of course the second distinction plays its main role here. I must add that the philosophical uses of the clusters are not unrelated. Some philosophers, for instance, maintain that an existent is what is or could be named (respectively denoted, designated...) by a word or expression. When we wish to speak without indicating some specific ontological commitments, we could avoid all these verbs and borrow instead Frege's expression *standing for*. Following this use, the philosophers just mentioned could say that a name is a word or an expression that stands for an existent.

Frege's Realism

I WOULD LIKE now to briefly outline different positions one might take in the debate whether Frege is or is not a realist, or whether he is or is not a nominalist. Putting correctly these labels is not an easy thing, and a lot of ink was spread out in this direction. In this section, my intention is only to offer a concise presentation of this debate.

Was Frege a nominalist? Was he a Platonist? There are some passages in his writings that seem to demand affirmative answers to both questions. Of course this possibility of understanding him in both ways appears as an open debate among Frege's scholars. One well-known form of the debate is given by Dummett's vs. Sluga's way of reading Frege. Michael Dummett has tried to motivate Frege's realism by interpreting it as a reaction against Hegelian idealism,[2] and his position is very clearly affirmed and sustained in all his writings concerning Frege: "Frege's realism was not the most important ingredient in his philosophy: but the attempt to interpret him otherwise than as a realist leads only to misunderstanding and confusion."[3]

An irreconcilable position is firmly sustained by Hans Sluga. He holds that for Frege abstract objects were not real, and so the only ontological label, which fits him, is nominalist. Sluga's point of view pays also attention to take into account the historical framework in which Frege did his philosophical work. He interprets Frege in Lotze–Kant's traditional line of reasoning.[4]

Another form of the debate 'whether or not Frege is a nominalist' can be found in some papers by Bergmann, Klemke, Grosmann, Jackson and Caton.[5] The main debate is between the first two commentators, the rest of them are only shedding some useful light in the discussion. Gustav Bergmann maintained that Frege's ontology is that of a hidden nominalist. In a critical discussion Klemke argued that Frege was a strict realist. Reinhardt Grossmann then replied and maintained that, in effect, both of them were right. Since the term 'realism' may mean different things, in one sense Frege may be realist, whereas in another important sense, he may not. Howard Jackson and Charles Cato have brought up points which are related to the discussion, and in a latter paper Bergmann again took up the topic (along with others) and held that Frege is not a "dead-end nominalist," but "his nominalistic tendency is as pronounced as it could be." After much further effort to make sense of Frege's ontological position, Klemke argued again that, in the most customary meaning of term, Frege was a Platonist. But before trying to offer some answers to the questions involved in that debate, let us go first to see how some ontological terms appear in Frege's philosophy.

Frege's Account of *Is, Existence, Being*...

As Leila Haaparanta has pointed out,[6] one of the doctrines that Frege emphasizes in his writings is the thesis that words for being, such as the English word 'is,' are ambiguous. A large part of his philosophy can be seen as an attempt to make us realize the importance of keeping the different meanings of 'is' apart and to spot the philosophical mistakes brought about by our failure to see the ambiguity. But how is the verb 'to be' ambiguous in Fregean logic? Frege distinguishes from each other the following meanings of 'is':

(1) The *is* of identity (Phosphorus is Hesperus; $a = b$)
(2) The *is* of predication (Frege is a computer; $F(a)$)
(3) The *is* of existence:
(i) expressed by means of the existential quantifier and the symbol for identity (God is; $(\exists x)(g = x)$), or

(ii) expressed by means of the existential quantifier and the symbol for predication (There are human beings/There is at least one human being; $(\exists x)H(x)$),
and

(4) The *is* of class-inclusion, the generic implication (A horse is a four-legged animal; $(x)(P(x)\supset Q(x))$.

As it is shown in the brackets, each putative meaning of *is* has its own logical formalization. Frege discusses the different meanings of *is* on various occasions, but he does not present all four meanings in any single text. His view of *is* can be put together using pieces taken mainly from works like "Dialog mit Pünjer über Existenz," "Die Grundlagen der Mathemathik," and "Über Begriff und Gegenstand."[7] There are also several remarks on the subject in the rest of his writings.

According to Frege's terminology, an object literally speaking 'falls under' (*fällt unter*) a first-order concept (*Begriff erster Stufe*), in which case we use the *is* of predication (copula), while a first-order concept 'falls in' (*fällt in*) a second-order concept (*Begriff zweiter Stufe*).[8]

Existence is, for Frege, a second-order concept. This view is most clearly expressed in Frege's discussion of the ontological argument for God's existence, and in his critique of Hilbert's program. As early as in the *Begriffsschrift* he hints at his account of the *is* of existence, but a detailed argumentation is presented only later in "Dialog mit Pünjer über Existenz."[9] Frege continues his discussion in *Gl*, where he writes: "existence is analogous to number. Affirmation of existence is in fact nothing but denial of the number nought. Because existence is a property of concepts the ontological argument for the existence of God breaks down."[10]

In "Über Begriff und Gegenstand" he argues along the same line: "I speak of properties asserted of a concept, and I allow that a concept may fall under a higher one. I have called existence a property of a concept. How I mean this to be taken is best made clear by an example. In the sentence 'there is at least one square root of 4,' we have an assertion, not about (say) the definite number 2, nor about -2, but about a concept, *square root of 4*; namely that is not empty."[11] Later, in a letter to Hilbert, Frege criticizes again inferences from essence to existence, which were used in the ontological argument. His view of being is reminiscent of Kant's position in *Kritik der reinen Vernunft*, where Kant states: "*Being* is obviously not a real predicate; that is, it is not a concept of something which could be added to the concept of a thing" (A598/B626). It is far from surprising that Frege's view of *is*, as not being a common first-order concept, originates in Kant's philosophy.

The Foundations of Arithmetic

After this preliminary discussion, I will try now to focus on the main topic of the presentation, namely on Frege's conception of number. The discussion will focus mainly on *Gl*. It was published in 1884, after *Bgs* and before *Gg*, and in a sense, it is the most philosophical of the three works. The intention of the book is quite clear from the very beginning, as long as its subtitle is: "A logico-mathematical enquiry into the concept of number." Keeping in mind that Frege is one of the main founders of the analytic tradition in Western philosophy, the strategy of the book is very simple and clear. The work has two main parts. In the first (negative) part, namely, the first three sections, the author provides a critical analysis of various numerical conceptions, and in the second (positive) part, namely, the last two sections, he presents his own view regarding the concept of number.

It is worth mentioning from the very beginning that in the introduction to the book Frege invokes three fundamental principles:

(1) always to separate sharply the psychological from the logical, the subjective from the objective;

(2) never to ask for the meaning of a word in isolation, but only in the context of a proposition;

(3) never to lose sight of the distinction between concept and object.

In addition to these principles, the author offers some specific requirements that should be satisfied by the definitions of the number. One of these is that the numbers, as defined, "should be adapted for use in every application made of number, even although that application is not itself the business of arithmetic" (*Gl*, §19).

The Status of Arithmetical Knowledge

Frege claims that all true arithmetic statements, namely, all true statements about the positive whole numbers 0, 1, 2, 3... and so forth, are analytic truths. This means not only that they can be proved using formal logical methods by resorting only to strictly arithmetical principles and concepts, but also that there is in fact no need to resort to any distinctively arithmetic first principles; the only principles required are those of logic. It is here that Frege goes beyond the Euclidian standards of rigor, aiming to eliminate entirely the need to resort to intuition. However, Frege did not claim that the whole of mathematics can dispense with intuition: in geometry, he left the door open to spatial intuitions.

In stating his position Frege uses terminology introduced by Kant, but interpreted in his own way. Kant had claimed that all mathematical truths (by which he understood those of geometry and arithmetic) are *synthetic a priori*. This classifies mathematical truths on the basis of two distinctions drawn by Kant, namely, that between *analytic* and *synthetic* judgments, and that between *a priori* and *a posteriori* truths.

In turn, there are four main types of truth for Frege (*Gl*, §3):

- a truth is *analytic* if its proof depends only on general logical laws and definitions;
- a truth is *synthetic* if its proof cannot be given without relying on truths from a particular science;
- a truth is *a priori* if its proof can be given from completely general laws, which themselves neither need nor admit of proof;
- a truth is *a posteriori* if its proof cannot be given without appealing to facts, that is, to unprovable and non-general truths that contain assertions about particular objects.

This characterization rules out the possibility of analytic a posteriori truths, since general logical laws and definitions are assumed to neither need nor admit of proof. In other words, analiticity implies apriority, but not vice versa. There remain, in turn, three possible combinations regarding the status of mathematical truth: (aa) *analytic a priori*, (sa) *synthetic a priori*, (sp) *synthetic a posteriori*.

The second position (sa), as I have already mentioned, is endorsed by Kant. He makes the distinctions between a *qualitative* logical reasoning, which is in accordance with formal logical principles, and a *quantitative* arithmetical reasoning, which means in fact calculating. While both kinds of reasoning are *a priori*, the first is analytic, while the second is synthetic. Obviously, however, that would be in contradiction with Frege's logicist project, namely the reduction of arithmetic to logic, and therefore it is inacceptable for him to support this variant.

The third position (sp) is endorsed by Mill, and it is harshly criticized by Frege. It is clear that the thesis that "all mathematical knowledge is empirical" could be not accommodated in Frege's system. Mill's definitions of numbers "are not definitions in the logical sense; not only do they fix the meaning of a term, but they also assert along with it an observed matter of fact" (*Gl*, §7).

The only possibility left is to maintain the first position, that of analytic a priori. This view was endorsed previously by Leibniz, who did not draw any distinction between the status of logical and mathematical laws. All of them are necessary truths based upon the principle of contradiction. It should also be noted that Leibniz's influence upon Frege is great and substantial. As we will see further on, Leibniz's fundamental insight, whereby every number can be defined in terms of its predecessor, offers the basis for Frege's solution regarding the concept of number.

What Numbers Are Not...

FOR FREGE, the truths of arithmetic are analytic a priori, they "govern all what is numerable. This is the widest domain of all; for it belongs not only the actual, not only the intuitable, but everything thinkable" (*Gl*, §14). This is also a reminiscent of Leibniz's position: "Some things cannot be weighed, as having no force and power, some things cannot be measured, by reason of having no parts; but there is nothing which cannot be numbered. Thus number is, as it were, a kind of metaphysical figure" (*Gl*, §24).

As Michael Beaney[12] states very explicitly, "in the second section Frege attacks two main positions regarding the nature of number: empiricism and psychologism. The empiricism he has in mind involves the conception of number as a property of external things. Once again, Mill is the main target, and Frege offers two reasons for not treating numbers in the same ways as qualities, whether primary qualities, such as solidity, or secondary qualities, such as color."

Firstly, Frege argues that qualities belong to external things "independently of any choice of ours," whereas what number we ascribe to something depends on our way of viewing it (*Gl*, §22). The *Iliad*, for example, can be thought of as one poem, or as 24 Books, or as some large number of verses. A pile of cards can be thought of as one pack, or as 52 cards, or 40 points in bridge. *One* pair of boots can be thought of as *two* boots (*Gl*, §25).

Secondly, Frege argues that the concept of number is applicable over a far wider range than color and solidity, and in particular, can be applied to what is non-physical (*Gl*, §24). On the other hand, if the number that can be ascribed to something depends on our way of viewing it, then it is tempting to regard such ascriptions, and the number itself, as purely subjective, like an idea or whatever. But to talk of ideas, understood as mental phenomena, is to suggest a psychological conception of number, and Frege is no less critical of this alternative position. Anti-psychologism remained one of the most dominant features of Frege's philosophy throughout his life. The principle that "there must be a sharp separation of the psychological from the logical, the subjective from the objective," as we have already seen, is the first of the three fundamental principles of the *Foundations*. For Frege, the realm of the psychological or subjective is the realm of ideas, understood as private mental entities. His fundamental objection to psychologism in logic and/or mathematics is that it rules out communication and makes arguments pointless. "If the number two were an idea, then it would straightaway be mine only. Another's idea is already as such another idea. We would then have perhaps many millions of twos. One would say: my two, your two, one two, all twos" (*Gl*, §27).

Thus, in spelling out the implications of constructing numbers as ideas, then, what Frege presents us with is in fact a *reductio ad absurdum* of psychologism. Then, numbers are objective. But then, are they like real objects? Here, Frege distinguishes what is objec*t*ive ('objektiv') from what is *actual* ('wirklich').[13] The real objects are 'actual,' in the sense that they are *handleable* ('handgreiflich') or *spatial* ('räumlich'), such that what is actual (the world of material substance) is only part of what is objective. We could say that numbers are "real" in the sense in which they are objective, namely, not mind-dependent entities. But for sure they are not actual. He explicitly says: "the axis of the earth is objective, so is the center of mass of the solar system, but I should not call them actual in the way the earth itself is so" (*Gl*, §26). We do, of course, speak of the equator as an *imaginary* line, but we do not mean by this that it is merely *imaginated*: "it is not a creature of thought, the product of a psychological process, but is only recognized or apprehended by thought" (*Gl*, §26).

By the end of the second section, then, both empiricism and psychologism have been rejected as offering viable accounts of arithmetic; however, we are left only with the positive assertion that numbers are objective, through non-actual. In the final part of the second section, Frege mentions one further position, namely, the set theory of numbers, which he understands as taking one of two forms: constructing numbers either as sets of objects or as sets of units. Neither view, remarks Frege, provides an account of the numbers 0 and 1, but the second view, he suggests, demands separate discussion, which he takes up in the third section. In fact, however, Frege's objections to both views are clarified in this part of the book.

Frege's analysis of numbers as sets[14] can be presented in the form of a dilemma. Either the elements of these sets are different (as they would be if they were different objects), or they are identical. If they are different, then the same problem arises as in the case of psychologism, namely, that there will be as many *twos* as there are different pairs of objects in the universe. But this first view, as we have already seen, could be easily demolished by *reductio ad absurdum*. Consequently, in the hope of avoiding this problem the second horn of the dilemma should be taken in turn, namely, constructing numbers as sets of units. The idea is simple and attractive, and this is in fact the intuition captured by the contemporary standard set theory. However, says Frege, if the units really are identical, then (so to speak) they would merge into one, and the whole theory collapses.

For example, let's start with the Leibnizian definition of 5:

$5 = 1 + 1 + 1 + 1 + 1$

According to the conceptions of numbers as sets of units, this is to be understood as follows:

$5 = \{1, 1, 1, 1, 1\}$

But if "1" is the name of an object, and each occurrence of "1" refers to the same object, then:

$5 = \{1\}$

Numbers, then, cannot be constructed for Frege just as sets (collections) of things, because for Frege, whether these things are identical with one another or not, absurdity results, and calling them "units" only serves to cover up the problem.[15]

Summing up, at the end of his negative critical analysis, we obtain at least some positive results concerning the nature of numbers:

(1) Arithmetical propositions are analytical a priori truths.

(2) Numbers are not properties of external things, since ascriptions of number depend on the concepts under which the things are classified.

(3) What is numerable is everything thinkable, not just the sensible or intuitable.

(4) Numbers are not subjective ideas, they are objective, though non-actual.

(5) Numbers cannot be constructed either as sets of objects or as sets of 'units.' The problem as to whether to treat units as the same or different shows that 'unit' ('Einheit') and 'one' ('eins') must be distinguished. But then, what are numbers?

What Numbers Are…

THE SECOND half of the book deals with the problem of defining the number in a very logical way. So, the Fregean task is now to formulate definitions, without presupposing a preliminary understanding of the expression "the number that belongs to the concept F," which provides a means of determining, for any *a* and *b*, whether $a = b$ or not. The strategy is to find a logically definable proposition, through which "to form the content of a judgment that can be constructed as an equation on each side of which is a number" (*Gl*, §63). The suggestion is to use the first of the following propositions to define the second:

(1) The concept F is equinumerous to the concept G (that means that there are as many objects falling under F as under G, or there are just as many Fs as Gs);

(2) The number of Fs is identical with the number of Gs (that means that the number that belongs to the concept F is the same as the number that belongs to the concept G).

The strategy is thus to define 'numerical identity' (*Gleichheit der Zahlen*) in terms of 'one-to-one correlation' (*beiderseits eindeutige Zuordnung*) or 'equinumerosity' (*Gleichzahligkeit*). Frege notes that this strategy "seems recently to have gained widespread acceptance amongst mathematicians. If a waiter wishes to be certain of laying exactly as many knives on the table as plates, he has no need to count either of them; all he has to do is to lay immediately to the right of every plate a knife, taking care that every knife on the table lies immediately to the right of a plate" (*Gl*, §70).

Frege takes his definition of identity from Leibniz's *salva veritate* substitution principle: those things are the same of which one can be substituted for the other without loss of truth. What Frege understands by this is what is often called *Leibniz's Law of Identity*, interpreted as comprising both the 'Principle of the Indiscernability of Identicals' (reading the equivalence from left to right) and the 'Principle of Identity of Indiscernibles' (reading the equivalence from right to left):

$(x = y) \leftrightarrow (\forall F)(Fx \leftrightarrow Fy)$

What this law provides is a definition of identity in purely logical terms (within second order predicate logic, where quantification over properties is allowed). It is this that justifies Frege in taking the concept of identity as already known. Frege goes on to remark that "in universal substitutability all the laws of identity are contained" (*Gl*, §65), and this is correct, so long as the substitutability is restricted to extensional contexts, something that Frege was only clear about later. Thus, we will have that:

(3) The number of Fs = the number of Gs iff there is a ono-one correspondence between the Fs and the Gs.

If two collections are said to be *equinumerous* whenever there is a one-one correspondence between them, then equinumerosity will be a relation between collections which can be shown to be an equivalence relation and which can be defined using only formal, logical notions. Therefore, numbers can be defined as equivalence classes under this equivalence relations, namely that:

(4) The number of Fs = (df) the class of classes X which are equinumerous with the class of Fs.

This gives us in a sense numbers and a definition of the concept "is a number":

(5) x is a number = (df) there is a concept F such that e is the number of Fs.

Still, it does not give us any object which might be a plausible candidate for the natural numbers. The task of locating these, however, has been reduced to that of discovering which of the numbers just defined should get the names 0, 1, 2… and so on. We would want it to be the case that:

0 = the number of Fs iff there are no Fs

1 = the number of Fs iff there is exactly one thing which is F

and so on…

Since the number of Fs is the class of all classes containing the same number of elements as the extension of Fs, all that is required is to be able, for each number, to pick up a class which has just that number of elements. This must be done in such a way that the membership of the selected class is logically determined. Here again Frege showed great ingenuity:

(6) 0 = (df) the number of objects x, such that $x \neq x$

The further definitions required are as follows:

(7) 1 = (df) the number of objects x such that 0 = x

(8) m = n + 1 (m immediately follows n in the numbers series) = (df) there is a concept F(x) such that m = the number of Fs & there is an object c such that F(c) & n = the number of objects y such that F(y) & $y \neq c$

The result is that the number 0 is the number that belongs to the concept *not identical with itself*. This can be further reformulated, in a manner that satisfies Frege's requirements for an explicit definition, as the number 0 is the extension of the concept "equinumerous to the concept *not identical with itself*."

Let's take now the concept *to be identical with 0*. Since one and only one object falls under that concept, namely, the number 0, the number that belongs to this concept is the number 1. The number that belongs to the concept *falling under the concept "identical with o" but not identical with 0*, on the other hand, is clearly 0, since nothing can fall under this concept. So we can conclude in such a manner that 1 is the successor of 0. The number 1 is the number that belongs to the concept *identical with 0*, or, better said, the number 1 is the extension of the concept "equinumerous to the concept *identical with 0*."

Further, we will have that the number 2 is the number that belongs to the concept *identical with 0 or 1*, or the number 2 is the extension of the concept equinumeorus to the concept *identical with 0 or 1*.

Furthermore, for any number, we will have that the number n + 1 is the number that belongs to the concept *member of series of natural numbers ending with n*.

Set & Extension

In the previous section, I presented a reconstruction of the Fregean strategy, using concepts like "set" and "extension." Since they cannot be found as such in the original text, I have to say why I resorted to these notions. As we already have seen, Frege is reluctant to reduce numbers to sets. However, I am using them, because I think that his account provides, if not an explicit, at least an implicit definition of a cardinal number as a class of equinumerous concepts.[16] This is an implicit and preliminary result, which is similar to the modern account of numbers, where numbers are reduced to sets. At this point, it is worth mentioning an interesting result provided by John Bell. He indicates how, along a Fregean line of thought, a general version of Zermelo's well-ordering theorem can be derived.[17] In another paper,[18] Bell continues by showing that there are some essential similarities among Frege's concept of 'number', Zermelo's notion of 'set' and 'von Neumann ordinals.' All these points support my decision to use the notion of class in this technical context, in spite of Frege's explicit reluctance.

Perhaps even more controversial might be the recourse to the technical Fregean notion of *extension*. The term it is not yet well-defined in *Gl*, so, I am using it in the sense established in later works.[19] As Demopoulos pointed out, "If in *Grundlagen* Frege expressed a certain indifference toward extensions of concepts, by the time of *Grundgesetze* he had come to regard their use as essential to the goal of showing the independence of arithmetic from intuition."[20] In *Grundgesetze* the 'extension' of a concept means the value range of the function which corresponds to the concept. Consequently, here 'extension' is used as referring to "the class of objects that fall under the concept." Together with the correlation between concepts and objects due to the Hume's principle and the context principle, this constitutes the grandeur and failure of the Fregean construction. Quite well known is the breakdown of Fregean system after Russell's attack of the *Basic Law V* of *Gg*. However, modern authors, like George Boolos and Crispin Wright, putting some restrictions, prevent the collapse of Frege's system, showing that the whole construction of it is consistent.[21] What is need in fact—since, due to Hume's principle, we correlate concepts with objects—is to be aware of the fact that concepts are in reality more numerous than objects, and they could not be put into a one-one relation (see also Cantor's diagonalization procedure).

Numbers As Objects & Context Principle

Two points still need to be mentioned here. Firstly, by virtue of the third principle, that everything is either a concept or an object, as long as we arrived at a definition of the number as being the extension of a concept, numbers are in turn objects, and that is what enables some commentators to sustain Frege's Platonism. Recall now his statement from §53, where he said: "existence is analogous to number. Affirmation of existence is in fact nothing but denial of the number nought. Because existence is a property of concepts the ontological argument for the existence of God breaks down." But since a property of a concept is also a property (a second order property, in fact), how could we accommodate these two accounts that numbers are both concepts and objects. Moreover, recall also the third already mentioned principle from the introduction to the book: "never to lose sight of the distinction between concept and object." The accounts seem to be mutually incompatible. The problem is similar to the one detected by Frege in his "On Concept and Object": "we are confronted by an awkwardness of language, which I admit cannot be avoided, if we say that 'the concept *horse*' is not a concept."[22] Thus the problem is linguistic and not conceptual. 'The concept *horse*' is a proper name, and thus has as its reference a definite object (a horse…), but not a concept. Similarly, 'the concept *number*' is not a concept, it refers to an object. Therefore, numbers are objects, and not concepts, as someone may be tempted to deduce from some unclear passages in Frege's works. But what kind of objects they are still remains an open question. We can conclude this problem by pointing out a passage from Michael Resnik: "In the *Grundlagen*, Frege was an ontological Platonist and an objective idealist. The application of the context principle to the analysis of number is a move within the rationalist tradition which seeks to show how our knowledge of arithmetic is based upon the faculty of reason."[23]

Secondly, since Frege assumes that he has shown that numbers are objects, they must be treated as such. But, since they are objects, a new problem arises by posing the Kantian question "How are numbers given to us?" According to Kant, objects can be given only trough sensible intuition. Frege, however, as we have already seen, rejected the notion that numbers are any kind of perceptible feature of things, or that numbers are objects of which we can have intuitions. The problem is therefore difficult, especially for someone influenced by Kant. His solution was to invoke his second principle, namely, the *context principle*: "only in the context of a sentence does a word have meaning." In this light, Frege converts the problem into an inquiry of how the senses of sentences containing terms for numbers are to be fixed. And this is for Michael Dummett[24] the 'linguistic turn' of philosophy, and that enables him to see Frege as the first analytic philosopher and, consequently, Frege's *Grundlagen* as the first true work of analytic philosophy.

It must be added that since in *Grundlagen* it is not clear what Frege means by "Bedeutung," there is room for two different interpretations of the context principle:

only in the context of a sentence does a word have a *sense*;

only in the context of a sentence does a word have a *reference*.

In distinguishing two readings of the context principle, I do not intend to claim that Frege maintained two versions of the principle. It does seem legitimate, however to ask whether Frege used his principle to deal with that aspect of a meaning of an expression, which he later called its *sense*, or with that aspect he later called its reference, or both. But this is another story, and I intend to tackle it in another paper.

Conclusion

Even if *Gl* contains some unclear remarks or passages, it remains, without doubt, one of the best argued pieces of philosophy. It is clear that it marks a new style of philosophizing, where arguments, good definitions and clear distinctions play a major role in this kind of enterprise. Each step further needs to be argued and conclusively proved. Something also new is the recourse to language, in order to solve theoretical problems. But beyond language stands 'reality,' and thus the ontological status of Fregean numbers is still in question. Personally, I incline towards endorsing a Platonist reading of Frege's philosophy. Besides the arguments presented above, there are two other possible arguments against interpreting Frege as a nominalist. The first is that his constant and strong rejection of *formalism* also endorses a realist reading of his philosophy, for he rejects formalism exactly because it does not explain the applicability of arithmetic to the real world. The second is the fact that, against *psychologism*, he keeps insisting on the objectivity of numbers, and thus it is difficult to see how Frege could coherently reject subjective contents, yet accept nominalist objective contents. Since numbers are not accepted to be present only *in mente*, then the most natural reading will be to accept them as being *in re* and not just *in vocis*, which expresses a lower ontological commitment.

So, the most natural and coherent position would be to interpret Frege as a realist. Numbers and senses may exist in a third realm, populated with Platonic entities, which make arithmetic and language objective and applicable to the real world.

❑

Notes

1. For more, see Bergmann (1968a), 42–43.
2. Dummett (1967), 225: "In a history of philosophy Frege would have to be classified as a member of the realist revolt against Hegelian idealism, a revolt which occurred some three decades earlier in Germany than in Britain."
3. Dummett (1993), 109.
4. Sluga (1993a), 139: "Michael Dummett, following an established line of reasoning, has interpreted Frege as a realist. But his claim that Frege was arguing against a dominant idealism is untenable. While there are passages in Frege's writings that seem to support a realistic interpretation, others are irreconcilable with it. The issue can be resolved only by examining the historical context. Frege's thought is, in fact, related to the philosophy of Hermann Lotze. Frege is best regarded as a transcendental idealist in the Lotze–Kant tradition."
5. All of them are generously collected in Klemke (1968c).
6. Haaparanta and Hintikka (1986a), 269: "What Frege and Russell accomplished was to make the ambiguity of 'is' a cornerstone of modern first-order logic."
7. I refer to Frege's main works using the following almost standard abbreviations: *Bgs* = *Begriffschrift*, *Gl* = *Grundlagen*, *Gg* = *Grundgesetze*, SB = "Über Sinn und Bedeutung." Also, by PW, PMC, CP and FR, I refer to standard English collections of Frege's works; see References.
8. See *On Concept and Object*, in CP, 51.
9. In PW.
10. *Gl*, §53, p. 65.
11. *On Concept and Object*, in CP, 49.
12. Beaney (1996), 86.
13. In German *wirklich* ('actual') comes from *wirken* ('to cause'). Thus, in order to be considered 'actual' something must have the power to cause effects. But this is not the case of the numbers, and, therefore, they are characterized as 'objective,' but 'non actual' things.
14. In fact, for Frege, numbers are conceived of as logical objects, i.e. as extensions of numerical concepts.
15. A modern and, somehow, similar account of the problem of reducing numbers to sets is provided by Paul Benacerraf in his paper "What Numbers Could Not Be." The author concludes that numbers are not objects, and, therefore, Platonism is rejected. The argument runs as follows. If natural numbers are mathematical objects, and if all mathematical objects are sets, then the problem is *which* sets the natural numbers are. According to one account, due to von Neumann, they are finite ordinals, while, according to Zermelo's account they are something different. Moreover, there seems to be no principled way to decide between the reductions. But then, what kind of entities numbers are?
16. See also Demopoulos (1995), 5.
17. For details, see ibid., 21–26.
18. See Bell (1995).

19. In "Sinn und Bedeutung," and, especially, in *Grundgesetze*.
20. Demopoulos (1995), 15.
21. For more see Heck (2011).
22. Frege (1960), 46.
23. Resnik (1980), 166.
24. Dummett (1991a), 111: "§62 is arguably the most pregnant philosophical paragraph ever written. It does not merely introduce the important notion of a criterion of identity, considered as associated with any proper name or other singular term: it is the very first example of what has become known as the 'linguistic turn' in philosophy. Frege's *Grundlagen* may justly be called the first work of analytical philosophy. . . . His solution was to invoke the context principle: only in the context of a sentence does a word have meaning. On the strength of this, Frege converts the problem into an enquiry how the sentences containing terms for numbers are to be fixed. *There* is the linguistic turn."

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Abstract
Frege on Numbers

The paper discusses Frege's concept of numbers as logical objects, focusing essentially on Frege's Platonism. While the issue could be approached from two perspectives, ontological and epistemological, and while both of the aforementioned points are connected and it is very hard to make a sharp distinction between them, at least in Frege's case, the present study pursues the first, ontological approach. As an explicit exposition of a Fregean ontology is not to be found in Frege's philosophy, the author seeks to infer his ontological commitment from other clear claims.

Keywords
Frege, philosophy of mathematics, nominalism, Platonism, realism, ontology