Is Mathematics the Theory of Instantiated Structural Universals?

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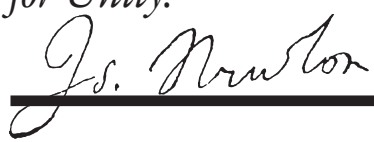
"By Number we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity."



As is well known, there are several ways in which a structuralist can reasonably argue for the mind-independent existence of mathematical objects. One may start, for example, with a holistic view of science and resort to indispensability arguments, as Michael Resnik does in his 1997 book *Mathematics as a Science of Patterns*. But one could also start from analytic metaphysics, take universals as primitive entities, and reduce mathematics to a theory about them. This is a strategy inspired by Armstrong's theory of instantiated structural universals, developed in his 1978 book *Universals and Scientific Realism*, and adopted by John Bigelow in his 1988 book *The Reality of Numbers*.

I will not discuss in this paper indispensability arguments for realism

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about mathematical objects, since many such discussions are already available in the literature.[1] Instead, what I want to do is argue that one cannot defend realism about numbers on the basis of a metaphysical realism about instantiated structural universals. More generally, I also want to suggest that it is misleading to take a metaphysical view as a basis for the ontology and epistemology of mathematics. In order to support my argument, and motivate my suggestion, I will discuss what I take to be Bigelow's failed attempt to reduce number theory to a metaphysical theory about instantiated structural universals (henceforth, SUs). This attempt fails, I argue, because it purports to reduce a theory we understand pretty well (number theory) to one which one can barely make sense of (the theory of SUs), but also and perhaps more importantly because it flies in the face of solid mathematical knowledge.

The outline of my paper is as follows. First, I briefly present Armstrong's theory of structural properties as instantiated universals and Lewis's devastating criticism of this theory. Afterwards, I argue that several responses to this criticism, by Armstrong, Bigelow, and more recently, by Joan Pagès, can hardly succeed. Finally, I show that one possible construal of structural universals via non-well-founded sets is resisted by the realist structuralist about mathematics, and conclude by pointing out an issue that would have to be addressed by anyone who wants to pursue Bigelow's reductionist project: the alleged countability of the real numbers.

Armstrong's Structural Universals and Lewis' Criticism

ARMSTRONG DEFINES structural properties in the following manner: "A property, S, is structural iff proper parts of particulars having S have some property or properties, T… not identical with S, and this state of affairs is, in part at least, constitutive of S" (Armstrong 1978, 69). Then he adds that "a structural property must be complex." One motivation for Armstrong's introduction of structural properties as instantiated universals is to allow for the possibility that there are no simple universals, i.e., the possibility that our world displays an "infinite descending complexity" of properties and relations (see Armstrong 1997, 32f.; compare also Williams 2007). Armstrong considers, for example, that it might very well be the case that a particular has an infinity of (structural, and so, complex) properties (Armstrong 1978, 72). But he also notes that the notion of SUs might lead to a composition problem. For instance, the property *being two electrons* involves, in a certain sense, the property *being one electron*. The latter has to be taken twice over, and this is problematic since as a universal *being one electron* is one, not many (Armstrong 1978, 70).

Armstrong's discussion gave rise to a debate that began with a detailed analysis of SUs by David Lewis in his 1986 paper "Against Structural Universals." Lewis notes that a structural universal can be instantiated by many particulars, and therefore can be located in different spatiotemporal positions. Where it is instantiated, it is wholly located as a nonspatiotemporal part of a particular. Further, any particular that instantiates it has to have proper parts and display the following characteristic: "there is a necessary connection between the instantiating of the structural universal by the whole and the instantiating of other universals by the parts" (Lewis 1986, 27). His examples of SUs are those instantiated by polymers: *methane* (CH_4), *butane* ($CH_3CH_2CH_2CH_3$), *isobutane* ($CH_3CH(CH_3)_2$), etc. These involve other universals like *carbon*, *hydrogen*, and the relation of *bonding*.

Lewis takes himself to undermine all reasons for believing in SUs, except the one suggested by Armstrong, which I have already mentioned above: the need to allow for the possibility of infinite descending complexity. But he notes that infinite complexity can be more successfully handled by class nominalism and by trope theory (Lewis 1986, 31). I want to focus, in the remainder of this section, on his threefold analysis of SUs.

According to Lewis, an SU can be conceived of (i) linguistically, i.e., as a set whose elements are themselves (simple) universals, (ii) pictorially, i.e., as a whole, mereologically composed out of other universals taken as proper parts, or (iii) magically, i.e., as a whole, non-mereologically composed out of other universals. All these conceptions purport to explain the entailment relation between instantiations of SUs and instantiations of the other universals involved.

(i) explains SUs as relations between a set and its members. Instantiating, for example, the SU *methane* (i.e., a set) is clearly impossible without also instantiating the universal *carbon* (i.e., a member of the set). (ii) explains SUs as relations between a whole and its proper parts. It seems pretty clear that an SU cannot be wholly instantiated without its proper parts being also instantiated. (iii) gives no explanation of the entailment relations. Under this view, the SU *methane* is completely distinct from the universal *carbon*, but their instantiations are nevertheless in a necessary part-whole relation and their being so remains unexplained.

Now, Lewis criticizes all these three ways of conceiving of SUs. Let me briefly explain why he does so. (i) is rejected mainly because it seems to ignore the motivation for believing in SUs in the first place—allowing for the possibility of infinite complexity. Since a set has to be constructed out of non-set elements, the "linguistic" conception seems to assume the existence of simple universals. Of course, one can ask why would a set have to be constructed out of non-set elements? Lewis seems to take here into account only set theories that contain an axiom of foundation, like ZFC. But surely one can discard this axiom and ac-

cept non-well-founded sets (i.e., sets that do not have any non-set element) at no consistency costs. More on this below.

The "magical" conception, (iii), is rejected because it makes no sense of composition, other than a metaphorical one. Since an SU is mereologically atomic, i.e., it has no proper parts, there can be no non-magical explanation of the necessary connection between, e.g., the instantiations of *butane* and the instantiations of *carbon* or *hydrogen* or *bonding*. This necessary connection is simply taken as a "brute modal fact" (Lewis 1986, 41f.). But, on the one hand, the theory of SU is supposed to be modal free, since according to its own defenders, it should be the basis for an account of modality.[2] On the other hand, to say that *carbon* and *hydrogen*, once combined, disappear in *methane*, or that the latter as a partless whole is still somehow composed of the former, is not to provide any such explanation.

Lewis focuses on the shortcomings of the "pictorial" conception (ii) and makes four attempts to fix it. Under this conception, the SU *butane* is isomorphic to a butane molecule, and the parts of the SU are themselves universals instantiated by the parts of the butane molecule. An SU is thus said to be "mereologically composite." (ii) appears to successfully explain the necessary connection between the instantiating of *butane* and the instantiating of *carbon* and *hydrogen* and *bonding* as a connection between a whole and its proper parts. Furthermore, it also seems to allow for the possibility of infinite complexity. However, there is still the composition problem already mentioned above: "Each methane molecule has not one hydrogen atom but four. So if the structural universal *methane* is to be an isomorph of the molecules that are its instances, it must have the universal *hydrogen* as a part not just once, but four times over. Likewise for *bonded*, since each molecule has four bonded pairs of atoms. But what can it mean for something to have a part four times over? What are there four of? There are not four of the universal *hydrogen*, or of the universal *bonded*; there is only one" (Lewis 1986, 34). Hence, it seems that, after all, the mereological composition of universals is sheer mystery. How does Lewis attempt to solve the composition problem and thereby fix the "pictorial" conception?

Lewis' first proposal is to drop the isomorphism between the SU and its instances (Lewis 1986, 36). An SU would be mereologically composed of its parts each taken only once. So, for example, *butane* would be composed of *carbon*, *hydrogen*, and *bonded*. But this obviously does not work, since then there would be no difference between *butane* and *methane*, both being composed of the very same three parts. His second proposal is to insist that two SUs could be mereologically composed of the very same parts, but that the difference consists in how many times over some part is taken in each of them. So, for example, in *butane*, *carbon* is taken 4 times over, *hydrogen* 10 times over, and *bonded* 13 times

over, while in *methane*, *carbon* is taken only once, *hydrogen* 4 times, and *bonded* is also taken 4 times over. Therefore, *butane* is different from *methane*. However, this does not work either, since then one cannot make any difference between *butane* and *isobutane*.

The third attempt to fix the "pictorial" conception of SUs drops mereological composition and postulates instead a "principle of uniqueness": no combining operation can yield different results if the items operated upon are the same (Lewis 1986, 38). But if several operations were applied in different orders, then one could get different results even if one started with the same universals. Thus, we could construct different SUs even if all we started with was *carbon*, *hydrogen*, and *bonded*. The problem with this fix is that, as Lewis points out, the notion of a nonmereological composition seems to be oxymoronic. There can very well be nonmereological combining operations, but those can hardly be taken as compositions. (In the next section, I analyze Armstrong's opposite view, i.e., his endorsement of nonmereological composition, and I will show how it leads him into trouble.)

Lewis' fourth proposal preserves the isomorphism between an SU and its instantiations, but postulates the existence of duplicate universals, e.g., the four occurrences of *hydrogen* in *methane*. Lewis calls them "amphibians," since they are universals that somehow duplicate themselves, so there is not only one universal for a given predicate, but many. Thus, *butane* has 10 *hydrogen* amphibians, 13 *bonded* amphibians, and 4 *carbon* amphibians. Of course, questions arise immediately as to the cogency of this proposal. As Lewis points out, we were interested in explaining the relationship between the SU *methane* and the universal *carbon*, not the connection between *methane* and the amphibian *carbon*. And if the amphibian *carbon* is all that is involved in the SU, then what about the universal *carbon* itself?

Several Responses to Lewis' Criticism

In his reply to Lewis, in the same issue of the *Australian Journal of Philosophy*, Armstrong attempts to make sense of nonmereological composition by first providing a counterexample to the above "principle of uniqueness": two distinct states of affairs, *a*R*b* and *b*R*a*, are composed of the same parts, two individuals, *a* and *b*, and a non-symmetrical relation, R (Armstrong 1986, 85).[3] He identifies SUs with *types* of states of affairs (Armstrong 1997, 34f.). So, for example, *methane* will be the *type* of a particular complex state of affairs, which is mereologically composed of nine simpler states of affairs (i.e., 4 hydrogen atoms, 4 bindings, and one carbon atom).

The immediate problem with this idea is that, although an SU is supposed to be, according to Armstrong, a nonspatiotemporal part of its instantiations, no account is given of how the *type* of a state of affairs can be a nonspatiotemporal part of its *tokens*.[4] It also seems that no such account can be given since the *type-token* relation is rather different from the part-whole relation. Furthermore, one would have to explain the connection between the *type* of the complex state of affairs and the *types* of the simpler states of affairs involved. And here one will stumble again upon the same problems raised by Lewis: how can one take the *type* hydrogen four times over, when there is only one such *type*? One would eventually have to postulate the existence of some other amphibians, which are both *types* and *tokens* at the same time. Quite a conundrum.[5]

Bigelow's first response to Lewis' rejection of the "pictorial" conception is to bite the bullet and admit amphibians in his ontology (Bigelow 1986, 95). Bigelow considers *methane* to be mereologically composed of four amphibians, each containing the universal *hydrogen* as a part. However, he does not offer any single argument to support this view, and it is pretty clear that the same composition problem as above still comes up. For the universal *hydrogen* still has to be taken four times over, even if now each time as a part of an amphibian. In addition, being part of an amphibian must be itself an amphibian, and so, it seems that universals not only are contained in amphibians, but they also instantiate amphibians. But since an amphibian is in fact many, the question arises, which amphibian does a universal really instantiate?

Bigelow's second and more articulate response is inspired by his views on quantity and begins by distinguishing three theoretical levels (Bigelow and Pargetter 1989, 5f.):

level-one: material individuals (hydrogen atoms, methane molecules, etc.);
level-two: properties and relations between level-one material individuals (being hydrogen, being methane, being bonded, etc.); these may be first order, or possessed simpliciter, and second order, or possessed in virtue of other properties or relations;
level-three: 'numerical' relations that hold between level-two properties or relations (being co-instantiated, having the same number of instantiations, etc.).

Now, on the basis of this scheme of three levels, *methane* is taken as a (conjunctive) level-two, second order, relational property, i.e., as having a *part* which contains the universal *carbon*, having a *part* which contains the universal *hydrogen*, having yet another *part* which contains the universal *carbon*, and yet another which contains the universal *hydrogen*, and so on. That is, something is a methane

molecule in virtue of having parts which instantiate (amphibian) *parts* and contain other molecules instantiating universals (*carbon*, *hydrogen*, etc.). According to Bigelow, it is not the case that the SU *methane* is composed mereologically by its conjuncts, but what is essential (i.e., constitutive) to the SU *methane* is that it stands in a level-three, 'numerical' relation with its (nine) conjuncts: "Thus, 'being methane' is so related to 'being carbon' that 'being methane' cannot exist without standing in that relation to 'being carbon'" (Bigelow and Pargetter 1989, 7). On this picture, then, the necessary connection between instantiations of *methane* and instantiations of *carbon* or *hydrogen* is not to be explained by mereology, but by reference to these constitutive level-three relations.

However, I think that this solution does not work, either. For how can one explain the fact that one constitutive level-three relation between *methane* and *hydrogen* is distinct from the other three constitutive level-three relations between *methane* and *hydrogen*, since there is only one *methane* and only one *hydrogen*?[6] Furthermore, the level-three relations are themselves universals, and they obviously cannot be simple. But if they are complex, then their structure has to be explained, too. So, I think that Bigelow's second response to Lewis does nothing more than pushing the problem of composition one step back.

More recently, Joan Pagès has also attempted to defend the theory of SUs from Lewis' criticism. She believes that one can make sense of the mereological composition of SUs, if one takes them to be constituted by two properties, *carbon* and *hydrogen*, and two relations, *bonded* and "a certain formal relation holding between instantiations of those universals" (Pagès 2002, 219). Like Bigelow's own proposal, this one, too, departs from Armstrong's vision of a sparse theory of universals, but also from a common conception of universals (see Pagès' own comments on page 220, footnote 6). Pagès' account is, nevertheless, able to distinguish mereologically between *butane* and *isobutane*,[7] because even if no difference would be apparent when considering only the simpler universals *hydrogen*, *carbon*, and *bonded*, there are other distinct universals that enter mereologically in the composition of *butane* and *isobutane*: the formal relations between the instantiations of these simpler universals. But the mystery of mereological composition of SUs is still there, I believe, because no explanation is provided of how one universal, e.g., *hydrogen*, can be taken four times over in the composition of *methane*. What is the use, then, of being able to make a difference between two isomorphic SUs, when there does not seem to be any way of making good sense of the mereological composition of a single SU taken separately?

Bigelow's Reduction of Numbers to Instantiated Structural Universals

I HAVE SHOWN in the previous two sections that the theory of SUs hardly makes any sense and that Lewis' criticism of it appears to stand unrefuted. This immediately suggests that it is quite unreasonable to even attempt to reduce mathematics to this theory. However, one way out for the reductionist might be indicated by a closer consideration of Lewis' rejection of the "linguistic" conception of SUs. As we have seen in section 1 above, this conception is rejected because it seems to assume the existence of simple universals (i.e., non-set elements of sets) and, thus, to block the possibility of infinite complexity. But the assumption could be dropped, as I already suggested, by construing SUs within a non-standard set theory, i.e., by discarding the axiom of foundation and accepting only non-well-founded sets (i.e., only sets that do not have any non-set element). The reduction of number theory to the theory of SUs would then of course count as a set-theoretical reduction, a very convoluted one to be sure, due to the existence of self-membered sets and universal sets (which can be mathematically accommodated though).[8]

Nonetheless, the reductionist himself resists this reduction as "too bold" (Bigelow 1986, 96), and as driven by nominalist motivations (Bigelow 1988, 46), and states quite bluntly: "I am not defending the view that a number is a set: I am arguing neither that a number is a set of n-membered sets, nor that it is a paradigm n-membered set. Nor, even, is it a property of n-membered sets. Rather, the number n is a universal which is instantiated by the members of any n-membered set" (Bigelow 1988, 56). And further: "numbers are not themselves sets. Rather, they are universals which are instantiated by sets, amongst other things" (ibid., 116). More precisely, taking into account the distinction between the levels presented above in section 2, natural numbers are level-two relations, whereas rational, real, imaginary, and complex numbers are level-three relations.

So let us adopt, for *reductio*, the view that numbers are instantiated structural universals. My goal, in doing so, is to show how this view leads to mathematical absurdities. In order to do that, let us first consider a difficulty raised by Peter Milne about the issue of locating relations spatiotemporally.

Like Lewis, Milne finds structural universals "a huge mystery" (Milne 1994, 309). He takes up the contention, which supports Bigelow's views on natural numbers, that level-two relations (i.e., relations between physical individuals) could be said to be wholly located within any region which contains those individuals (Bigelow 1988, 22) and are, therefore, themselves physical. He correctly replies that this is a *non sequitur*: "From the assertion that the only relations

there are obtain between physical things it does not follow that the relations themselves are physical and hence locatable in space-time" (Milne 1994, 311). Milne suggests that Bigelow's position requires a problematic commitment to two theses: (a) all relations between physical things are physical, and (b) anything physical is (with approximation) locatable in physical space.

In fairness to Bigelow, one must note that, in his view, the question "Where is it?" is not such that it admits of a straightforward answer. Rather, he is willing to loosen the localizability requirement for physicalism and, in view of quantum mechanical limitations, to talk about electrons as (being closer to) universals (Bigelow 1988, 25). Hence, thesis (b) can be attributed to him only if the approximation of locality in physical space is sufficiently large to accommodate multi-locatable physical things (e.g., relations, numbers, etc.). Be that as it may, one would still need an argument in support of thesis (a), but Bigelow simply provides none.

As proposed, let us provisionally adopt the view defended by Bigelow and then briefly look at the immediate consequences. According to this view, natural numbers are level-two relations between physical relata. What about rational and real numbers? Bigelow quotes from Newton's *Arithmetica Universalis*: "By Number we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity." He reads Newton's ratio between quantities as a proportion (understood in the most general sense), or a level-three relation between universals (Bigelow 1988, 69). Thus, rational numbers are level-three relations of this sort. Furthermore, a real number is said to be a level-three relation between two incommensurable magnitudes, i.e., between two universals. The square root of 2, for example, is the relation between the universal *side* and the universal *diagonal* in the SU *square*. By thesis (a) above and because according to Bigelow's Armstrongian position all universals are physically instantiated, it follows that this relation is physically instantiated. Thus, real numbers are physically instantiated. But then, since according to him everything physical can be counted, it follows that real numbers are countable (ibid., 116). Needless to say, this flies in the face of solid mathematical knowledge (Cantor's diagonal proof of the uncountability of the reals, in particular).

In conclusion, I believe it is plausible to regard Bigelow's structuralism about numbers, defended on the basis of Armstrong's metaphysical realism about universals, as untenable. Numbers are not instantiated structural universals. What this suggests, more generally, is that taking a metaphysical view as a basis for a philosophy of mathematics may be misleading. But as I pointed out at the outset, there are other routes open to the realist structuralist about mathematics.

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Notes

1. See for example, Colyvan 2003 and Baker 2003.
2. "Modality is a huge mystery to me, and I would love to have a clearer understanding of it. One of my key motives for exploring metaphysics and mathematics has been the desire to get a clearer grip on modality. My hunch is that modal claims rest on facts about relations among universals" (Bigelow 1988, 38). And further: "I hope that an adequate theory of universals should give an explanation of the grounds for *necessity* and *possibility*. . . . the construction of universals, using sets and possibilia, is beside the point from the perspective I am taking here" (103).
3. It is essential to Armstrong's point that R be a non-symmetrical relation. So, for example, one could say that Mary, John, and loving can combine to form two states of affairs, John's loving Mary and Mary's loving John. But, of course, this is still not sufficient to show that the nonmereological composition of SUs makes sense, since no support is given to the idea that they involve non-symmetrical relations.
4. Similar comments have been recently made in Pagès 2002, 217.
5. For another attempt to deflect Lewis' criticism and further develop Armstrong's conception of structural universals, see Kalhat 2008.
6. Note that it would not help to give here Armstrong's counterexample, mentioned above at the outset of section 2, since non-symmetry is not an obvious characteristic of the level-three relations constitutive of *methane*.
7. Her graphical representation of an isobutane molecule (Pagès 2002, 216) is, however, incorrect since it instantiates *bonded* 15 times, and so makes *isobutane* different from *butane* even without the help of formal relations.
8. For other recent attempts to overcome the problems pointed out by Lewis, cf. Mormann 2010.

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Abstract

Is Mathematics the Theory of Instantiated Structural Universals?

The paper contends that one cannot defend realism about numbers on the basis of a metaphysical realism about instantiated structural universals, suggesting that it is misleading to take a metaphysical view as a basis for the ontology and epistemology of mathematics. The author criticizes Bigelow's attempt to reduce number theory to a metaphysical theory about instantiated structural universals, which purports to reduce number theory to the theory of structural universals, and which flies in the face of solid mathematical knowledge. The study begins with a presentation of Armstrong's theory of structural properties as instantiated universals and of Lewis's devastating criticism of this theory, arguing that several responses to this criticism, by Armstrong, Bigelow, and more recently, by Joan Pagès, can hardly succeed. Finally, it contends that one possible construal of structural universals via non-well-founded sets is resisted by the realist structuralist about mathematics. The conclusion highlights an issue that would have to be addressed by anyone who wants to pursue Bigelow's reductionist project: the alleged countability of the real numbers.

Keywords

philosophy of mathematics, number theory, structural universals, real numbers, D. M. Armstrong, D. Lewis, J. Bigelow